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Eigenstates of boson creation operator

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Abstract. We examine the existence of right-hand eigenstates (or eigenkets) of the boson creation operator $a^\dagger$ and determine their coordinate and their Bargmann representation. The eigenkets of the creation operator are ultrasingular and cannot be considered as a limiting case of normalizable states. Applications of these eigenstates as auxiliary states for purposes of representation of states by path integrals over coherent states are discussed. A completeness relation for coherent states on paths through the complex plane is derived and special examples of its application are considered.

PACS. 42.50.Dv Nonclassical field states; squeezed, antibunched, and sub-Poissonian states; operational definitions of the phase of the field; phase measurements

1 Introduction

It is well-known [1,2] that the coherent states are the eigenkets of the boson annihilation operator a. Although nonorthogonal to each other for different eigenvalues but complete (and even overcomplete), the set of coherent states plays an outstanding role in the development of quantum optics since the sixties of past century and found many important applications. As to the question whether the creation operator $a^\dagger$ does possess (right-hand) eigenstates or not, there are different kinds of answers: the usual one is that $a^\dagger$ has no normalizable eigenstates [3] which found introduction into monographs of quantum optics, *e.g.* [4] and textbooks of quantum mechanics, *e.g.* [5] (Chap. V, Sect. 32). This, in principle, right answer is easily understood and can be proved but it is only half the truth. The other possible answer is that $a^\dagger$ possesses nonnormalizable right-hand eigenstates which play an important role for auxiliary purposes although they are not realizable in physical devices [6–11]. Then it is the problem to construct these eigenstates in explicit way and to demonstrate their potential capabilities for applications. In [12], the problem of the eigenstates of the creation operator is solved for the paraparticle (parabose and parafermi) case and is applied to obtain resolutions of unity by contour integrals.

The main purpose of the present paper is the derivation of the eigenstates of the boson creation operator in different representations (Fock-state representation, Bargmann representation, position and momentum representation) and to show their role as auxiliary states for the problem of representation of arbitrary states by path integrals over coherent states. The importance of these representations is the possible reduction of many linear physical transformations of a state (*e.g.*, time evolution processes, amplification and attenuation, damping or decay) to the superposition of the corresponding simpler transformations of the coherent states. The derivations to the eigenvalue problem, we make in two essentially different approaches. In Section 2, we make the derivation from the recurrence relations for the coefficients in the Fock-state expansion, where we have to apply known relations for multiplication of derivatives of the delta function with power functions. Before this, we show in Section 1, how plausible looking considerations can lead to the conclusion of the nonexistence of these eigenstates. In Section 3, we start from the eigenvalue problem for arbitrary linear combinations of a boson annihilation and creation operator and derive the eigenstates of the creation operator as an ultrasingular limiting case. This derivation possesses the advantage that it sets the eigenstates of the creation operator in relation to a very large class of states which are the normalizable squeezed coherent states with the coherent states as one limiting case and their nonnormalizable extensions comprising the eigenstates of the canonical operators Q and P. In Section 4, we give the explicit form of the representation of states by path integrals over coherent states and extend this in natural way to single and

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double path integral representation of operators. As examples, we consider the path integral representation of the annihilation and creation operator. In application to the density operator of a state this leads to the possibility of representation of mixed states by double path integrals over coherent states.

Let us first consider usual arguments for the impossibility of normalizable eigenkets of the boson creation operator. Suppose $a^\dagger$ possesses eigenstates $|z\rangle_\#$ to complex eigenvalues z according to

$$a^\dagger |z\rangle_\# = z\,|z\rangle_\#\,, \tag{1}$$

where the subscript "#" indicates that the state concerned belongs to $a^\dagger$ (in [12] is used instead the superscript "$'$"). Then in the coordinate representation $\langle q|z\rangle_\#$, we have with $Q \to q, P \to -\mathrm{i}\partial/\partial q$ (we set $\hbar = 1$)

$$z\,\langle q|z\rangle_\# = \langle q|\,a^\dagger\,|z\rangle_\# \;=\; \frac{1}{\sqrt{2}}\,\langle q|\,(Q - \mathrm{i}P)\,|z\rangle_\#$$
$$= \frac{1}{\sqrt{2}}\left(q - \frac{\partial}{\partial q}\right)\langle q|z\rangle_\#\,. \tag{2}$$

With the initial condition $\langle 0|z\rangle_\# = f(z)$, this leads to the solution

$$\langle q|z\rangle_\# = \exp\left(\frac{q^2}{2}\right)\exp\left(-\sqrt{2}zq\right)f(z), \tag{3}$$

which, obviously, is nonnormalizable as function of q. These arguments can also be obtained in the following modified way. Using the completeness of the number basis $|n\rangle$, one has

$$|z\rangle_\# = \sum_{n=0}^{\infty}|n\rangle\,\langle n|z\rangle_\#\,. \tag{4}$$

As a result of $\langle n|a^\dagger = \sqrt{n}\,\langle n-1|$, one derives in connection with (1) the recurrence relations

$$0 = z\,\langle 0|z\rangle_\#\,,\quad \langle 0|z\rangle_\# = z\,\langle 1|z\rangle_\#\,,$$
$$\sqrt{2}\,\langle 1|z\rangle_\# = z\,\langle 2|z\rangle_\#\,, ...,$$
$$\sqrt{n}\,\langle n-1|z\rangle_\# = z\,\langle n|z\rangle_\#\,,\quad n = 1,2,... \tag{5}$$

This can lead to the (incorrect) conclusion that for $z \neq 0$, all coefficients $\langle n|z\rangle$ have to be vanishing but then remains the case $z = 0$. For example, Davydov pointed out that according to (5), if $z = 0$, we have $\langle n|z\rangle_\# = 0$ for all n from the second and the following equations in (5). If $z \neq 0$, it follows from first equation in (5) that $\langle 0|z\rangle_\# = 0$. Hence from (5) it can be seen that $\langle n|z\rangle_\# = 0$ is still true for $n = 1,2,...$ that means for all n. Thus it is asserted that $|z\rangle_\# = 0$ and no eigenstate of $a^\dagger$ exists. However, the above reasoning is actually not rigorous since the equation which looks like $zf(z) = 0$ in form has the solution $f(z) = c\delta(z)$ for real z with arbitrary coefficient c (*e.g.*, [13]). So we should find the eigenfunctions of the creation operator $a^\dagger$ in δ-function form but the situation is more complicated when z are arbitrary complex numbers. In this case, one has to look to $\delta(z)$ as to an analytic functional over a basis space of entire functions which guarantees the necessary extension of the generalized function $\delta(z)$ to a linear functional over the complex plane.

Due to its high singularity, the eigenkets of the creation operator are not realizable states in physical devices. Nevertheless, they play an important role as auxiliary states for decompositions of normalizable states into path and contour integrals over coherent states [6–12]. In this regard the situation is similar to the eigenstates of the position and momentum operator which lead to the representation of states by wave functions. However, the eigenstates of the creation operator are "more singular" compared with the eigenstates of position and momentum operator and cannot be normalized by means of the delta function. In this regard, we have here another situation as in the mentioned case. All these states can be embedded into a larger class of states which are the eigenstates of arbitrary linear combinations $a + \zeta a^\dagger$ of an annihilation and creation operator [9]. For $|\zeta| < 1$, the eigenstates of $a + \zeta a^\dagger$ are normalizable squeezed coherent states. For $|\zeta| = 1$, the eigenstates of $a + \zeta a^\dagger$ are identical with the eigenstates of the rotated canonical operator $R(\varphi)Q(R(\varphi))^\dagger$ with $R(\varphi) \equiv \exp(\mathrm{i}\varphi a^\dagger a)$ the rotation operator and are normalizable by means of the delta function (weakly nonnormalizable states). In the special case $\varphi = 0$ this leads to the eigenstates $|q\rangle$ of the operator Q and in case $\varphi = \pi/2$ to the eigenstates $|p\rangle$ of the operator P. Finally, for $|\zeta| > 1$, the eigenstates become ultrasingular and make (with appropriate standardization) in the limiting case $|\zeta| \to \infty$ the transition to the eigenstates of the creation operator (strongly nonnormalizable states). This is represented more in detail in Section 3 after having developed the Fock-state representation of the eigenkets of the creation operator in a more elementary way.

2 Fock-state expansion and Bargmann representation of eigenkets of the creation operator

In this section, we derive first the Fock-state expansion of the eigenstates of the creation operator $a^\dagger$ in analytic form which we denote by $|(z)\rangle_\#$

$$|(z)\rangle_\# = \sum_{n=0}^{\infty}|n\rangle\langle n|(z)\rangle_\#,\quad \frac{\partial}{\partial z^*}|(z)\rangle_\# = 0. \tag{6}$$

The recurrence relations (see Eq. (5)) to this ansatz can be written in the form

$$0 = z\,\langle 0|(z)\rangle_\#\,,\quad \langle 0|(z)\rangle_\# = \frac{z^n}{\sqrt{n!}}\langle n|(z)\rangle_\#, \tag{7}$$

and can be solved that leads to the following Fock-state expansion of the eigenkets of the creation operator [6–12]

$$|(z)\rangle_{\#} = \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n!}} \delta^{(n)}(z)|n\rangle, \tag{8}$$

where $\delta^{(n)}(z)$ denotes the nth derivative of $\delta(z)$. For the purpose of proof, we can use the identity $z\delta^{(n)}(z) = -n\delta^{(n-1)}(z)$ or, more generally,

$$z^l \delta^{(n)}(z) = (-1)^l \frac{n!}{(n-l)!} \delta^{(n-l)}(z), \qquad l, n = 0, 1, ..., \tag{9}$$

for arbitrary nonnegative integer l and n (proof, *e.g.*, [14]). We have set the complex variable z in $|(z)\rangle_{\#}$ into round brackets to show that the states in this standardization depend on z in analytic way without dependence on the complex conjugated variable z^*. The solution (8) can also be represented in the following form

$$|(z)\rangle_{\#} = \exp\left(-a^{\dagger} \frac{\partial}{\partial z}\right) \delta(z)|0\rangle \cdot \tag{10}$$

We now derive the Bargmann representation of the states $|(z)\rangle_{\#}$ and introduce the notations

$$|(z)\rangle \equiv \exp\left(za^{\dagger}\right)|0\rangle = \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}}|n\rangle,$$

$$\langle(z^*)| \equiv \langle 0|\exp\left(z^* a\right) = \sum_{n=0}^{\infty} \frac{z^{*n}}{\sqrt{n!}}\langle n|. \tag{11}$$

The states $|(z)\rangle$ and $\langle(z^*)|$ are the analytic coherent states without the normalization factor $\exp\left(-zz^*/2\right)$ present in the usual (normalized) coherent states $|z\rangle$ that is shown again in the notations by setting z and z^* into circular brackets (sometimes, analytic coherent states are denoted by $\|z\rangle$ that, however, becomes nonunique and unfavourable in scalar products). Furthermore, deviating from Dirac's convention for bra-states, we denote the adjoint states $\langle(z^*)|$ with the genuine variable z^* on which they depend in analytic form. For the scalar products of the eigenkets (10) with the analytic coherent states (11), we find

$$\langle(z)|(z')\rangle_{\#} = \delta(z - z'). \tag{12}$$

This is the Bargmann representation [1,2,15] of the eigenkets of the creation operator and it is in considered case not a usual entire function such as for normalizable states but a generalized function in form of an analytic functional. One can look to (12) also as to a biorthogonality relation. The coherent states as eigenstates of a non-Hermitian operator are nonorthogonal to each other for different eigenvalues but together with the eigenstates of the creation operator they form a biorthogonal (or dual) system of states normalized by means of the delta function.

The generalized functions $\delta^{(n)}(z - z')$ (derivatives of $\delta(z - z')$) are defined as linear functionals $\left\langle\delta^{(n)}(z - z'), \varphi(z)\right\rangle$ over a certain space of entire basis functions $\varphi(z) \in \mathcal{Z}$ corresponding formally to integrals over paths $\mathcal{P}$ through the complex plane in following way

$$\begin{aligned}\left\langle\delta^{(n)}(z - z'), \varphi(z)\right\rangle &\equiv \int_{\mathcal{P}} \mathrm{d}z \delta^{(n)}(z - z')\varphi(z) \\ &= (-1)^n \varphi^{(n)}(z'). \end{aligned} \tag{13}$$

Now, by using the representation by path integral, we find from equations (11, 12)

$$\begin{aligned}\int_{\mathcal{P}} \mathrm{d}z|(z)\rangle\,{}_{\#}\langle(z)| &= \int_{\mathcal{P}} \mathrm{d}z \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(-1)^n z^m \delta^{(n)}(z)}{\sqrt{m!n!}}|m\rangle\langle n| \\ &= \sum_{n=0}^{\infty} |n\rangle\langle n| = I, \end{aligned} \tag{14}$$

where I is the identity operator in the Hilbert (Fock) space. With ${}_{\#}\langle(z)|$ we have denoted the adjoint state to $|(z^*)\rangle_{\#}$ in the following way (also deviating from Dirac's rule and thus writing the genuine complex variable on which it depends; we mention here that in [12] this problem is solved in a slightly different way)

$$\begin{aligned}{}_{\#}\langle(z)| &\equiv \langle 0|\exp\left(-a\frac{\partial}{\partial z}\right)\delta(z) \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n!}} \delta^{(n)}(z)\langle n|. \end{aligned} \tag{15}$$

Thus we have obtained a resolution of the identity by pairs of dual coherent states $|(z)\rangle$ or eigenkets of the annihilation operator and eigenbras ${}_{\#}\langle(z)|$ of the annihilation operator. This can be considered as a completeness relation of the coherent states over arbitrary paths through the complex plane. In (14), it is difficult to distinguish to which side the index "#" belongs. However, both variants

$$\int_{\mathcal{P}} \mathrm{d}z|(z)\rangle\left\{{}_{\#}\langle(z)|\right\} = \int_{\mathcal{P}} \mathrm{d}z\left\{|(z)\rangle_{\#}\right\}\langle(z)| = I, \tag{16}$$

are true and, in principle, this nonuniqueness even could remain open without leading to errors. In Section 4, we use the completeness relation (14) to get representations of states by path integrals over coherent states.

3 Eigenstates of creation operator as nonnormalizable ultrasingular squeezed coherent states

We now describe another very instructive way to the eigenkets of the creation operator. The right-hand eigenstates of the creation operator can be obtained as an ultrasingular limiting case of nonnormalized squeezed coherent states [9].

Squeezed coherent states can be introduced as solutions of the eigenvalue problem for an arbitrary linear combination of a boson annihilation and creation operator in the following form

$$(a+\zeta a^{\dagger})|(\alpha,\zeta)\rangle = \alpha|(\alpha,\zeta)\rangle, \qquad \alpha,\zeta\in \mathrm{C}, \tag{17}$$

to arbitrary complex eigenvalues α and with an arbitrary complex parameter ζ. The general solution of this eigenvalue problem can be represented in the following nonnormalized form by application of a nonunitary operator onto the vacuum state

$$|(\alpha,\zeta)\rangle \equiv \exp\left(\alpha a^{\dagger} - \frac{\zeta}{2}a^{\dagger 2}\right)|0\rangle \cdot \tag{18}$$

This can easily be verified by applying the operator identity (I is identity operator in Hilbert space of states)

$$\exp\left(-\alpha a^{\dagger} + \frac{\zeta}{2}a^{\dagger 2}\right)\left(a+\zeta a^{\dagger}\right)\exp\left(\alpha a^{\dagger} - \frac{\zeta}{2}a^{\dagger 2}\right) = a+\alpha I, \tag{19}$$

onto the vacuum state $|0\rangle$ using $a|0\rangle = 0$. The operator identity (19) can be derived as the special case $f(B) = B = a+\zeta a^{\dagger}$ of the following well-known operator identity for arbitrary operators A and B (sometimes called Baker-Campbell-Hausdorff formula but the genuine Baker-Campbell-Hausdorff formula concerns the much more entangled case of the product of two Lie group operators in the exponential mapping from Lie algebra to Lie group)

$$\exp(A)f(B)\exp(-A) = f\left(B + \frac{1}{1!}[A,B] + \frac{1}{2!}[A[A,B]] + ...\right). \tag{20}$$

From (18), using the generating function for Hermite polynomials $H_n(x)$, the following Fock-state representation of the states (18) can be obtained [9,16,17]

$$|(\alpha,\zeta)\rangle = \sum_{n=0}^{\infty} \frac{\left(\sqrt{2\zeta}\,\right)^n}{2^n\sqrt{n!}} H_n\left(\frac{\alpha}{\sqrt{2\zeta}}\right)|n\rangle \cdot \tag{21}$$

The parameter α is an arbitrary complex displacement parameter and ζ an arbitrary complex squeezing parameter. The states (18) are nonnormalized but depend in analytic way on both variables α and ζ. This was called the nonunitary approach to squeezed coherent states [16,17]. The adjoint states, we denote (again deviating from Dirac's rule) by the genuine analytic variables on which they depend as follows

$$\langle(\alpha',\zeta')| \equiv \langle 0|\exp\left(\alpha' a - \frac{\zeta'}{2}a^2\right). \tag{22}$$

By using the following generating function for products of two Hermite polynomials (formula of Mehler, [18] (Chap. 10.13))

$$\sum_{n=0}^{\infty} \frac{t^n}{2^n n!} H_n(x)H_n(y) = \frac{1}{\sqrt{1-t^2}} \times \exp\left\{\frac{2xyt - \left(x^2+y^2\right)t^2}{1-t^2}\right\}, \quad |t|^2 < 1, \tag{23}$$

we find for the scalar products of arbitrary squeezed coherent states (18)

$$\langle(\alpha',\zeta')|(\alpha,\zeta)\rangle = \frac{1}{\sqrt{1-\zeta\zeta'}} \times \exp\left\{\frac{2\alpha\alpha' - (\zeta'\alpha^2 + \zeta\alpha'^2)}{2(1-\zeta\zeta')}\right\}, \quad |\zeta||\zeta'| < 1, \tag{24}$$

and in particular for the scalar product of squeezed coherent states with themselves

$$\langle(\alpha^*,\zeta^*)|(\alpha,\zeta)\rangle = \frac{1}{\sqrt{1-\zeta\zeta^*}} \times \exp\left\{\frac{2\alpha\alpha^* - (\zeta^*\alpha^2 + \zeta\alpha^{*2})}{2(1-\zeta\zeta^*)}\right\}, \quad |\zeta|^2 < 1. \tag{25}$$

This last formula shows that the states $|(\alpha,\zeta)\rangle$ are normalizable and therefore physically realizable states only for $|\zeta| < 1$ and it gives the restriction for the possible choice of a normalization factor. However, formula (24) shows that the mutual scalar products of two states (18) exists (that means the sums in the Mehler formula are absolutely convergent) in every case when $|\zeta||\zeta'| < 1$. Therefore, we can give the states $|(\alpha,\zeta)\rangle$ a sense for auxiliary purposes if $|\zeta| \geq 1$. We mention here the following. In the more common unitary approach to squeezed coherent states, there is mostly used another complex squeezing parameter $\bar{\zeta}$ connected with ζ by

$$\zeta = \bar{\zeta}\frac{\mathrm{th}|\bar{\zeta}|}{|\bar{\zeta}|}, \quad \bar{\zeta} = \zeta\frac{\mathrm{Arth}|\zeta|}{|\zeta|} \quad \Rightarrow \quad |\zeta| = \mathrm{th}|\bar{\zeta}|, \quad |\bar{\zeta}| = \mathrm{Arth}|\zeta|. \tag{26}$$

Contrary to the parameter ζ, the mostly used squeezing parameter $\bar{\zeta}$ cannot be further extended to cases corresponding to $|\zeta| \geq 1$ because $|\zeta| \to 1$ corresponds to $|\bar{\zeta}| \to \infty$ and it can only be applied for normalizable squeezed coherent states. Furthermore, we mention that the coherent component of the squeezed coherent states given by the expectation value $\overline{a}$ of the annihilation operator a is related to the parameters α and ζ by

$$\overline{a} \equiv \frac{\langle(\alpha^*,\zeta^*)|a|(\alpha,\zeta)\rangle}{\langle(\alpha^*,\zeta^*)|(\alpha,\zeta)\rangle} = \frac{\alpha-\zeta\alpha^*}{1-\zeta\zeta^*}, \qquad \alpha = \overline{a} + \zeta\overline{a}^*. \tag{27}$$

Problems of the transition from nonunitary to unitary approach and *vice versa* are dealt with in detail in [17] and for our present aim, we cannot longer stay here with them.

We now divide the eigenvalue equation (17) with the solution (18) by ζ and substitute $\alpha = \zeta z$. After this it takes on the form

$$\left(\frac{1}{\zeta}a + a^\dagger\right)\exp\left(\zeta z a^\dagger - \frac{\zeta}{2}a^{\dagger 2}\right)|0\rangle = z\exp\left(\zeta z a^\dagger - \frac{\zeta}{2}a^{\dagger 2}\right)|0\rangle\cdot \quad (28)$$

If we multiply this equation by the complex factor $\sqrt{\zeta/(2\pi)}\exp\left\{-(\zeta/2)z^2\right\}$, we can write it in the form

$$\left(\frac{1}{\zeta}a + a^\dagger\right)\sqrt{\frac{\zeta}{2\pi}}\exp\left\{-\frac{\zeta}{2}\left(zI - a^\dagger\right)^2\right\}|0\rangle = z\sqrt{\frac{\zeta}{2\pi}}\exp\left\{-\frac{\zeta}{2}\left(zI - a^\dagger\right)^2\right\}|0\rangle\cdot \quad (29)$$

This equation is appropriate to make the limiting transition $\zeta \to \infty$ leading to

$$a^\dagger|(z)\rangle_\# = z|(z)\rangle_\#, \quad (30)$$

where $|(z)\rangle_\#$ is defined by

$$|(z)\rangle_\# \equiv \lim_{\zeta\to\infty}\sqrt{\frac{\zeta}{2\pi}}\exp\left\{-\frac{\zeta}{2}\left(zI - a^\dagger\right)^2\right\}|0\rangle = \delta\left(zI - a^\dagger\right)|0\rangle\cdot \quad (31)$$

By Taylor series expansion of $\delta\left(zI - a^\dagger\right)$ in powers of the creation operator $a^\dagger$, we find then

$$|(z)\rangle_\# = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\delta^{(n)}(z)a^{\dagger n}|0\rangle = \sum_{n=0}^{\infty}\frac{(-1)^n}{\sqrt{n!}}\delta^{(n)}(z)|n\rangle\cdot \quad (32)$$

This is identical with (10) and (11). Summarizing equations (29–32), the eigenkets $|(z)\rangle_\#$ of the creation operator can be considered as the following limiting case of nonnormalized squeezed coherent states $|(\alpha,\zeta)\rangle$ in the nonunitary approach

$$|(z)\rangle_\# = \lim_{\zeta\to\infty}\sqrt{\frac{\zeta}{2\pi}}\exp\left(-\frac{\zeta}{2}z^2\right)|(\zeta z,\zeta)\rangle\cdot \quad (33)$$

It does not play a role in which sector of the complex plane ζ goes to infinity because the result of the limiting procedure is considered as an analytic functional which belongs to the space $\mathcal{Z}'$ of generalized functions over the space $\mathcal{Z}$ [13].

The eigenkets $|(z)\rangle_\#$ of the creation operator are strongly nonnormalizable. There is yet another special case of the states $|(\alpha,\zeta)\rangle$, where they are individually nonnormalizable but become mutually normalizable by means of the delta function. We call such states weakly nonnormalizable. This is the case $|\zeta| = 1$ and it leads to the eigenstates of the canonical operators Q and P and, more generally, to the eigenstates of the rotated canonical operator Q that means of $Q(\varphi) \equiv R(\varphi)Q\left(R(\varphi)\right)^\dagger$ with $R(\varphi) \equiv \exp(\mathrm{i}\varphi a^\dagger a)$ the rotation operator. If we substitute $\zeta = |\zeta|\mathrm{e}^{\mathrm{i}2\varphi}$, then for $|\zeta| = 1$, we can write the eigenvalue equation (17) in the form

$$\begin{aligned} Q(\varphi)\big|(\alpha,\mathrm{e}^{\mathrm{i}2\varphi})\big\rangle &\equiv \frac{1}{\sqrt{2}}\left(\mathrm{e}^{-\mathrm{i}\varphi}a + \mathrm{e}^{\mathrm{i}\varphi}a^\dagger\right)\big|(\alpha,\mathrm{e}^{\mathrm{i}2\varphi})\big\rangle \\ &= \frac{1}{\sqrt{2}}\mathrm{e}^{-\mathrm{i}\varphi}\alpha\big|(\alpha,\mathrm{e}^{\mathrm{i}2\varphi})\big\rangle \\ &\equiv q\big|(\alpha,\mathrm{e}^{\mathrm{i}2\varphi})\big\rangle\cdot \end{aligned} \quad (34)$$

Therefore, we find

$$Q(\varphi)\Big|\left(\sqrt{2}\mathrm{e}^{\mathrm{i}\varphi}q,\mathrm{e}^{\mathrm{i}2\varphi}\right)\Big\rangle = q\Big|\left(\sqrt{2}\mathrm{e}^{\mathrm{i}\varphi}q,\mathrm{e}^{\mathrm{i}2\varphi}\right)\Big\rangle\cdot \quad (35)$$

This means that the eigenstates of $Q(\varphi)$ are proportional to the states $\Big|(\sqrt{2}\mathrm{e}^{\mathrm{i}\varphi}q,\mathrm{e}^{\mathrm{i}2\varphi})\Big\rangle$ and the proportionality factor can be used for a standardization. In particular, for $\varphi = 0$, we obtain the eigenstates $|q\rangle$ of the canonical operator Q and for $\varphi = \pi/2$ the eigenstates $|p\rangle$ of the canonical operator P (we renamed $Q(\pi/2) \to P$ and $q \to p$ in last case). We can check by their scalar products or by using the generating function for Hermite polynomials applied to the exponential operators that the usual standardization corresponds to

$$\begin{aligned} |q\rangle &= \frac{1}{\pi^{\frac{1}{4}}}\exp\left(-\frac{q^2}{2}\right)\exp\left(\sqrt{2}qa^\dagger - \frac{1}{2}a^{\dagger 2}\right)|0\rangle, \\ |p\rangle &= \frac{1}{\pi^{\frac{1}{4}}}\exp\left(-\frac{p^2}{2}\right)\exp\left(\mathrm{i}\sqrt{2}pa^\dagger + \frac{1}{2}a^{\dagger 2}\right)|0\rangle\cdot \end{aligned} \quad (36)$$

This, together with (11), leads to the following wave functions (in generalized sense) of the states $|(z)\rangle_\#$ in position and momentum representation

$$\begin{aligned} \langle q|(z)\rangle_\# &= \frac{1}{\pi^{\frac{1}{4}}}\exp\left(-\frac{q^2}{2}\right)\frac{1}{\sqrt{-2\pi}}\exp\left\{\frac{1}{2}\left(z - \sqrt{2}q\right)^2\right\}, \\ \langle p|(z)\rangle_\# &= \frac{1}{\pi^{\frac{1}{4}}}\exp\left(-\frac{p^2}{2}\right)\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2}\left(z + \mathrm{i}\sqrt{2}p\right)^2\right\}\cdot \end{aligned} \quad (37)$$

It agrees in the form with (3), where the chosen standardization of $|(z)\rangle_\#$ allowed to determine the function $f(z)$ explicitly. The eigenstates $|q\rangle$ correspond to squeezing parameter $\zeta = 1$ and the eigenstates $|p\rangle$ to squeezing parameter $\zeta = -1$. More generally, positive squeezing parameter $\zeta = \zeta^* > 0$ correspond to squeezing in direction of the coordinate q (abscissa) and negative squeezing parameter $\zeta = \zeta^* < 0$ to squeezing in direction of the coordinate p (ordinate). The longer axis of the squeezing ellipse is in the first case in p-direction and in the last case in q-direction [16,17]. In the general case of complex ζ, the squeezing direction is not identical with one of the coordinate directions.

Coherent states as eigenstates of the annihilation operator correspond to squeezing parameter $\zeta = 0$ within the class of squeezed coherent states $|(\alpha,\zeta)\rangle$ in the nonunitary approach and eigenstates of the creation operator as has been shown to the ultrasingular limiting case of parameter $\zeta \to \infty$. It is clear that one could form pairs of normalizable squeezed coherent states $|(\alpha,\zeta)\rangle$ and nonnormalizable squeezed coherent states $|(\alpha',\zeta')\rangle$ with $\zeta\zeta' = 1$ which form dual pairs and are appropriate to formulate completeness relations on paths through the complex plane for expansions by integrals over squeezed coherent states. We do not develop this here in detail because the most important case is the expansion in coherent states.

Thus we have shown that the nonunitary approach to squeezed coherent states provides besides the usual normalizable squeezed coherent states very important classes of nonnormalizable states which play a role as auxiliary states for different expansions of states. In case of the eigenstates $|q\rangle$ and $|p\rangle$ of the canonical operators Q and P, this is well-known, whereas in case of the eigenstates $|z\rangle_{\#}$ this is little known up to now. The approach to the problem of eigenstates of the creation operator *via* the ultrasingular limiting case of nonnormalized (analytic) squeezed coherent states is an alternative approach to the direct solution of this problem given in Section 2 and provides a deeper insight into the problem with new perspectives.

4 Path integrals over coherent states

Using the completeness relation (14), we may obtain a representation of arbitrary states $|\psi\rangle$ by path integrals over coherent states in the following way

$$|\psi\rangle = \int_{\mathcal{P}} \mathrm{d}z |(z)\rangle \, {}_{\#}\langle (z)|\psi\rangle \cdot \tag{38}$$

Thus we have to calculate the scalar products ${}_{\#}\langle (z)|\psi\rangle$ of the state under consideration with the eigenstates of the creation operator which play a role as auxiliary states for obtaining the representation by a path integral over analytic coherent states $|(z)\rangle$. We now consider examples.

For normalized coherent states $|\alpha\rangle$, we find

$$\begin{aligned} |\alpha\rangle &= \exp\left(-\frac{\alpha\alpha^*}{2}\right)\exp\left(\alpha a^\dagger\right)|0\rangle, \\ {}_{\#}\langle (z)|\alpha\rangle &= \exp\left(-\frac{\alpha\alpha^*}{2}\right)\delta(z-\alpha), \end{aligned} \tag{39}$$

and the coherent state is represented by a delta function of the complex variable which is an analytic functional. Therefore, an arbitrary finite superposition of coherent states (Schrödinger cat state) provides a path integral representation with an integral kernel in form of a sum over delta functions with support at the positions of the coherent states and with corresponding coefficients

$$\begin{aligned} |\psi\rangle &= \sum_{k=1}^{n} \lambda_k |\alpha_k\rangle, \quad \sum_{k=0}^{n} |\lambda_k|^2 = 1, \\ {}_{\#}\langle (z)|\psi\rangle &= \sum_{k=0}^{n} \lambda_k \exp\left(-\frac{\alpha_k\alpha_k^*}{2}\right)\delta(z-\alpha_k). \end{aligned} \tag{40}$$

As a more general example compared with (39), we consider squeezed coherent states $|(\alpha,\zeta)\rangle$ in the nonunitary approach which has been discussed in preceding section for other purpose and which are defined by (18) or (21). According to (38), we have to calculate their scalar products with the eigenstates of the creation operator that can be accomplished in the following way

$$\begin{aligned} {}_{\#}\langle (z)|(\alpha,\zeta)\rangle &= \langle 0|\exp\left(-a\frac{\partial}{\partial z}\right)\exp\left(\alpha a^\dagger - \frac{\zeta}{2}a^{\dagger 2}\right)|0\rangle\delta(z) \\ &= \langle 0|\exp\left(\alpha a^\dagger - \frac{\zeta}{2}a^{\dagger 2} + \zeta a^\dagger\frac{\partial}{\partial z}\right) \\ &\quad \times \exp\left(-a\frac{\partial}{\partial z}\right)|0\rangle \\ &\quad \times \exp\left(-\alpha\frac{\partial}{\partial z} - \frac{\zeta}{2}\frac{\partial^2}{\partial z^2}\right)\delta(z) \\ &= \frac{1}{\sqrt{-2\zeta\pi}}\exp\left(\frac{(z-\alpha)^2}{2\zeta}\right)\cdot \end{aligned} \tag{41}$$

In the first step, we changed the order of operations in the scalar product using the operator identity (20) with $f(B) = \exp(B)$ and $A = -a(\partial/\partial z)$, $B = \alpha a^\dagger - (\zeta/2)a^{\dagger 2}$. The sequence of "box-in-box" commutators $B, [A,B], [A[A,B]], ...$ terminates after the third term for the considered special operators. The operator which acts in (41) onto the delta function is a convolution operator which leads to the result written in the last line. This can be proved, for example, by Fourier transformation and its inversion.

By inserting (41) into (38), we obtain the following representation of (nonnormalized) squeezed coherent states by path integrals over nonnormalized (analytic) coherent states

$$|(\alpha,\zeta)\rangle = \frac{1}{\sqrt{-2\zeta\pi}}\int_{\mathcal{P}} \mathrm{d}z \exp\left(\frac{(z-\alpha)^2}{2\zeta}\right)|(z)\rangle \cdot \tag{42}$$

The integration path $\mathcal{P}$ through the complex plane is widely deformable but has to begin for convergence of the integral in infinity of one sector where the Gaussian function $\exp\left\{(z-\alpha)^2/(2\zeta)\right\}$ vanishes and has to end in the opposite sector. As mentioned in Section 3, for squeezing in q-direction (longer axis of squeezing ellipse in p-direction), we have $\zeta = \zeta^* > 0$ and we can choose an integration path parallel to the y-axis through the point $z = \alpha$ of the variable $z = x + \mathrm{i}y$ that leads to the specialization of (42)

$$|(\alpha,\zeta)\rangle = \frac{1}{\sqrt{2|\zeta|\pi}}\int_{-\infty}^{+\infty} \mathrm{d}y \exp\left(-\frac{y^2}{2|\zeta|}\right)|(\alpha+\mathrm{i}y)\rangle, \quad \zeta = \zeta^* > 0. \tag{43}$$

For squeezing in p-direction (longer axis of squeezing ellipse in q-direction), we have $\zeta = \zeta^* < 0$ and we can choose an integration path parallel to the x-axis through the point $z = \alpha$ of the variable $z = x + \mathrm{i}y$ that leads to the specialization of (42)

$$|(\alpha,\zeta)\rangle = \frac{1}{\sqrt{2|\zeta|\pi}} \int_{-\infty}^{+\infty} \mathrm{d}x \exp\left(-\frac{x^2}{2|\zeta|}\right) |(\alpha+x)\rangle, \quad \zeta=\zeta^*<0. \quad (44)$$

The weight functions are in both cases normalized Gaussian functions [19,20]. This can easily be understood in an illustrative way as weighted superpositions of circles on a straight line standing for the coherent states and leading to ellipses with the longer axis along this line and standing for the squeezed coherent states. However, the more general formula (42) admits deformations of the most simple integration paths chosen in (43) and (44). The left-hand sides of equations (42–44) can be converted to squeezed coherent states in the unitary approach [9,17]. However, this destroys a little the simplicity and harmony of the representations (42–44) because this is a nonanalytic approach and then the complex conjugated variables α^* and ζ^* to α and ζ appear additionally in the formulae. The same is if we use the usual normalized coherent states $|z\rangle$ in the path integral representations.

As a further example of state representation, we consider Fock states $|n\rangle$. From (15), we find

$$_{\#}\langle(z)|n\rangle = \frac{(-1)^n}{\sqrt{n!}}\delta^{(n)}(z), \quad (45)$$

and the path integral (38) takes on the form

$$\begin{aligned} |n\rangle &= \int_{\mathcal{P}} \mathrm{d}z \frac{(-1)^n}{\sqrt{n!}}\delta^{(n)}(z)|(z)\rangle \\ &= \frac{1}{\sqrt{n!}} \int_{\mathcal{P}} \mathrm{d}z \delta(z) \frac{\partial^n}{\partial z^n}|(z)\rangle \\ &= \frac{1}{\sqrt{n!}} \left\{ \frac{\partial^n}{\partial z^n}|(z)\rangle \right\}_{z=0}. \quad (46) \end{aligned}$$

The correctness of this representation of Fock states can be directly checked from the definition of the analytic coherent states $|(z)\rangle$ in (11). In case that the Fock-state expansion of a considered state is known, by using backwards the origin of the right-hand side of (46) from a path integral, it should be possible to reconstruct the path integral representation of the considered state.

The completeness relation in the two variants (16) can be used to obtain operator representations by path integrals over nonnormalized eigenstates of the annihilation and creation operator. The boson annihilation and creation operator themselves can be represented by

$$a = \int_{\mathcal{P}} \mathrm{d}z\, z|(z)\rangle\, _{\#}\langle(z)|, \quad a^\dagger = \int_{\mathcal{P}} \mathrm{d}z\, z|(z)\rangle_{\#}\, \langle(z)|. \quad (47)$$

For the density operator ϱ, one can find 4 possible representations by double path integrals over analytic coherent states, for example

$$\varrho = \int_{\mathcal{P}} \mathrm{d}z \int_{\mathcal{P}'} \mathrm{d}z' |(z)\rangle \left(_{\#}\langle(z)|\varrho|(z')\rangle_{\#} \right) \langle(z')|. \quad (48)$$

As was discussed in [21], this representation is related to the complex P-representation of Drummond and Gardiner [22] which is a special case of a class of generalized P-representations.

5 Conclusion

We have derived the right-hand eigenstates of the creation operator and have used them to formulate a completeness relation for coherent states on arbitrary paths through the complex plane. The highly singular eigenkets of the creation operator are dual (or biorthogonal, more specificly) to coherent states as the eigenkets of the annihilation operator. They play an important role as auxiliary states which are necessary to derive the explicit form of the expansion of arbitrary states by path integrals over coherent states. The introduction of the eigenstates of the creation operator as analytic functionals and of the coherent states as analytic functions of the complex variable z makes it possible to give a formulation where the integration paths can be deformed. On the other hand this means that the integration paths can be chosen in appropriate way depending on the physical content of the states as has been discussed for squeezed coherent states. The coherent states in the whole complex plane are overcomplete and one can select different complete subsets of coherent states as basis for the representation of arbitrary states. For example, the coherent states are also complete on a variety of contours in the complex plane. Using Cauchy's integral formula, this gives the possibility to derive representations of states by contour integrals over coherent states [6–12,23,24] that we did not discuss here. By means of the Stokes' formula and Green's function, one can establish the relations between representations by contour, area and path integrals [10]. Representations of states by path and contour integrals over coherent states are usually more illustrative than by integrals over the whole complex plane since the last are highly nonunique in their form due to the overcompleteness of the coherent states in the complex plane. This becomes obvious if we compare the corresponding representations for coherent states themselves which are delta-function kernels in path integrals over coherent states and Gaussian functions in phase-space integrals.

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